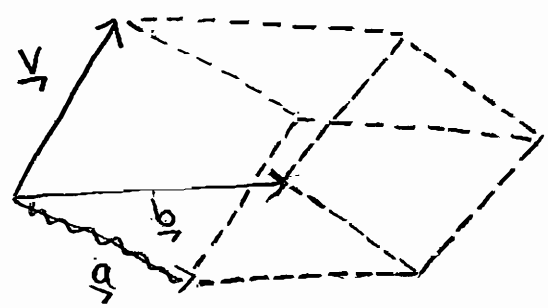
Flux in $\mathbb{R}^3$ [Andrew Critch, Math 53]

Consider an ordered parallelogram $\underset{\to}{a}, \underset{\to}{b}$, and some fluid flowing through $\underset{\to}{a}, \underset{\to}{b}$ at a constant velocity of $\underset{\to}{v}$. How much fluid flows out* of $\underset{\to}{a}, \underset{\to}{b}$ in one time unit?

Answer: $\boxed{\text{Signed Volume* of Figure}} = \boxed{\det(\underset{\to}{v}, \underset{\to}{a}, \underset{\to}{b})} = \boxed{\underset{\to}{v} \cdot (\underset{\to}{a} \times \underset{\to}{b})}$



* The order of $\underset{\to}{a}, \underset{\to}{b}$ determines which way is "out" by the right hand rule.

We define a "flux" operator "$\star \underset{\to}{v} \cdot$" which computes this:

$$(\star v \cdot)^{\text{of}} (\underset{\to}{a}, \underset{\to}{b}) := \boxed{\det(\underset{\to}{v}, \underset{\to}{a}, \underset{\to}{b})} = \boxed{\underset{\to}{v} \cdot (\underset{\to}{a} \times \underset{\to}{b})},$$

and read this as "the flux of $\underset{\to}{v}$ through $\underset{\to}{a}, \underset{\to}{b}$".

Remarks:

For those interested, $\star\underset{\rightharpoondown}{V}\cdot$ is called the "Hodge dual" of $\underset{\rightharpoondown}{V}$, and "$\star$" is called the "Hodge star" in higher math.

Recall that if, for example, $\underset{\rightharpoondown}{V} = 2\underset{\rightharpoondown}{i} + 3\underset{\rightharpoondown}{j} + 4\underset{\rightharpoondown}{k}$,

then $\underset{\rightharpoondown}{V}\cdot$ has a "life of its own" as $2dx + 3dy + 4dz$.

So does $\star\underset{\rightharpoondown}{V}\cdot$, but it involves the "wedge product," which is not part of this course:

$$\boxed{\star\underset{\rightharpoondown}{V}\cdot = 2dy \wedge dz + 3dz \wedge dx + 4dx \wedge dy}$$